Learning Assist Techniques for Differential Geometry

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出版者	法政大学大学院情報科学研究科
journal or publication title	法政大学大学院紀要. 情報科学研究科編
volume	13
page range	1-4
year	2017-03-31
URL	http://doi.org/10.15002/00021534

Learning Assist Techniques for Differential Geometry

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Abstract*— *In some areas of engineering, familiarity with classical differential geometry of curves and surfaces is preferable and related courses are included in the curriculum. This paper presents usage patterns of computer algebra systems to design an interactive mathematics class, taking up topics from differential geometry. Learning differential geometry of curves involves getting familiar with calculating tangents, normals, and binormals. The traditional approach of pencil and paper is still important. However, there is a common and typical drawback. Tractable exercise problems with pencil and paper are very difficult to find except for those in the classical textbooks, and they are very limited in number. We need more tractable exercise problems to allow the learners to iterate mathematical experiments until they become satisfied with their intuitive understandings. Further, mathematical experiments should be a kind of activity that keeps the learners motivated. To fulfil these requirements, we propose a new approach to these practices in classrooms and online courses. We present a special toolkit that assists the learners to gain geometrical insights through designing their own examples of curves and creating impressive animations. The toolkit also provides the teachers with facilities to prepare presentation materials for use in class.

I. INTRODUCTION

In some areas of engineering, such as computer graphics and image processing, familiarity with classical differential geometry of curves and surfaces is preferable, and related courses are included in the curriculum. This paper presents usage patterns of computer algebra systems to design an interactive mathematics class, picking up topics from differential geometry. Ohtake proposed a technique for implicit surface representation based on partition of unity, a very basic but formerly considered purely theoretical notion in differential geometry [1]. To be a good user of these techniques, or to further advance technology in those fields, we have to be familiar with basic notions in differential geometry. So, it is natural to include differential geometry courses in the curricula of engineering departments [2].

Learning differential geometry of curves involves getting familiar with calculating tangents, normals, and binormals with pencil and paper. Learners should carry out mathematical experiments with concrete examples as many times as they need to gain intuitive understanding of geometrical concepts such as curvature, torsion, and moving frames.

The traditional approach is still important in educational programs in science and any branches of engineering disciplines. However, there is a common and typical drawback. Tractable exercise problems with pencil and paper are very difficult to find except for in the classical textbooks, and they are very limited in number. We need more exercise problems to allow the learners to iterate mathematical experiments until they become satisfied with their intuitive understandings. Further, mathematical experiments should be a kind of activity that keeps the learners motivated.

To fulfill these requirements, we propose a new approach to these practices in classrooms and online courses. We present a special toolkit that assists the learners to gain geometrical insights through designing their own examples of curves and to create impressive animations. The toolkit also provides the teachers with facilities to prepare presentation materials for use in class.

Related work is published by Gray et. al [3], but our work is much more user interaction oriented.

Our method of introducing geometrical concepts depends heavily on computer calculus and animation with Mathematica. Students are asked to design a parametric curve with analytic expressions or possibly piecewise analytic ones. Then, they select an aircraft, which is a simple 3D-graphics object, chosen from the set of predesigned 3D objects. Next, with the aid of Mathematica, they calculate the tangent, normal, and binormal vector fields along the curve and let the aircraft fly along the curve with its nose pointing in the tangent direction, its wings kept parallel to the normal, the pilot's head pointing to the binormal direction. This way, they can feel in touch with tangents, normals, binormals, curvatures, and torsions.

Our toolkit consists of a collection of library functions and a set of instruction patterns in class. The toolkit gives firstly a set of ready-made materials for learners and teachers. It gives also a set of facilities for course designers to extend the collection of impressive teaching materials.

Supervisor: Prof. Shuichi YUKITA

This paper is organized as follows. In section 2, basic notions in differential geometry of curves are briefly reviewed. In section 3, instruction sequence patterns are described. In section 4, the design of flying objects is described. In section 5, working examples are given.

II. NOTIONS OF DIFFERENTIAL GEOMETRY OF CURVES

We briefly review the notions of tangent, normal, and binormal vector fields along a curve in Euclidean 3-space $\mathbf{R}^3$. Let $c(t)$ be a smooth curve in $\mathbf{R}^3$, where t runs through an appropriate closed/open interval on the real line $\mathbf{R}$. The tangent vector field of the curve is $c'(t)$. Normalizing it, we get the unit tangent vector field

$$e_1(t) = \frac{c'(t)}{|c(t)|}.$$

If we take its derivative, we get the normal vector field $e_1'(t)$. Normalizing it, we get the unit normal vector field

$$e_2(t) = \frac{e_1'(t)}{|e_1'(t)|}.$$

Then taking the cross product of $e_1(t)$ and $e_2(t)$, we get the binormal vector field

$$e_3(t) = e_1(t) \times e_2(t)$$

The triple $\langle e_1(t), e_2(t), e_3(t) \rangle$ is called the Frenet-Serret frame of the curve. In the following sections, we will see how students are instructed to do the due course of these calculations. For more details, see [4].

III. DESIGNING PROCESS

In this section, we describe a laboratory session for differential geometry of curves. Students are guided along the following scenario.

1. Design the trajectory along which the object fly (this trajectory can be any smooth curve; students can freely design parametric curves).
2. Calculate tangent, normal, and binormal vectors.
3. Design flying objects.
4. Observe flight simulation of flying objects.

In this chapter, we will describe steps 1 and 2.

A. *Design curve*

First, students design a curve. This curve should be defined parametrically. Parametric here means that it depends on a parameter (a variable with continuous values such as time and angle). For example, a student may design a curve as,

$$p(t) = (t\cos t, t\sin t, t),$$

where $0 < t < 8\pi$. Mathematica code is the List1.

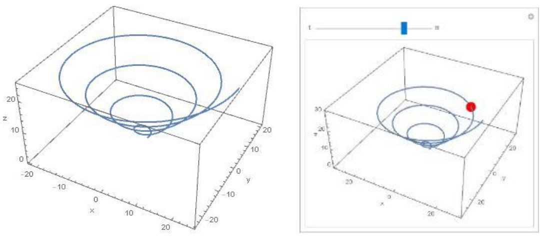

Fig. 1. parametric curve

List 1. Definition of a curve

```
p[t_] := {t Cos[t], t Sin[t], t}
```

Mathematica gives graphics output as the left one in Fig.1.

Next, the student traces trajectory with a simple object (like a ball). By doing this test, the learner can confirm that the trajectory that he or she designed is correct. See the right one in the captured images in Fig. 1.

The Mathematica code to produce interactive graphics in Fig. 1 is the following (List 2).

List 2. Test by a ball

```
Manipulate[
 Show [{g1, Graphics3D[{Red, PointSize
       -> 0.05, Point[p1[t]]}]},
       PlotRange -> {{-30, 30}, {-30,
       30}, {0, 30}}],
    {t, 0, 8 \[Pi]}]
```

B. *Tangent Vector Field*

After defining the curve, the student calculates the tangent vector field. The direction of this tangent vector represents the direction of travel of the flying object in the simulation to do from now. The tangent vector field is given as

$$\frac{dp}{dt}(t).$$

Students in mathematics department should first do it with pencil and paper. Anyway, Mathematica gives

List 3. Differential

```
D[p1[t], t]
   ==>{Cos[t]-t Sin[t],t Cos[t]+Sin[t],1}
```

Once the tangent vector field is obtained, the student is instructed to normalize it. Then, the student is instructed to define t_1 as the following.

$$\text{t1[t\_]} := \left\{ \frac{\text{Cos}[t] - t\,\text{Sin}[t]}{\sqrt{1 + (t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}}, \frac{t\,\text{Cos}[t] + \text{Sin}[t]}{\sqrt{1 + (t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}}, \frac{1}{\sqrt{1 + (t\,\text{Cos}[t] + \text{Sin}[t])^2 + (\text{Cos}[t] - t\,\text{Sin}[t])^2}} \right\}$$

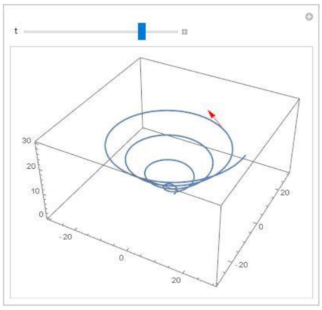

Fig 2. tangent represented by red arrow

To confirm the direction of the tangent vector, the student creates interactive 3D graphics of the moving vector. Captured images are in Fig. 2, where a red arrow represents the tangent vector.

The Mathematica code is the List4.

List 4. Confirm vector direction

```
Manipulate[
 Show[{g1,
   Graphics3D[{Red,
    Arrow[{p1[t], p1[t] + 15*t1[t]}]
    }]},
 PlotRange -> {{-30, 30},{-30, 30},
                {0,30}}],
 {t, 0, 8 \[Pi]}]
```

Normal Vector Field

The normal vector fields are given by differentiating the tangent vector. That is, it is the second differentiation of $p(t)$ as follows,

$$\frac{d^2}{dt^2}(p(t)).$$

After calculating this, students are instructed to normalize the result in the same way as with the tangent. Although these operations are simple and straight forward, it is very difficult to calculate with pencil and paper. CAS is in place. Then, students are instructed to define the unit normal vector field as follows.

$$\text{n1}[t\_] := \left\{\frac{-t\,(3+t^2)\,\text{Cos}[t] - (4+t^2)\,\text{Sin}[t]}{\sqrt{(2+t^2)\,(8+5\,t^2+t^4)}}, \frac{(4+t^2)\,\text{Cos}[t] - t\,(3+t^2)\,\text{Sin}[t]}{\sqrt{(2+t^2)\,(8+5\,t^2+t^4)}}, -\frac{t}{\sqrt{(2+t^2)\,(8+5\,t^2+t^4)}}\right\}$$

Students are instructed to do a test to look and feel the orientation of the normal vectors. The test image is in Fig. 3, where the tangent and normal vectors are shown in red and green, respectively.

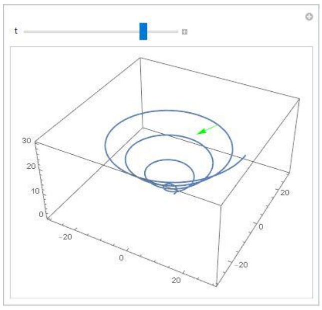

Fig. 3. normal vector represented by green arrow

C. Binormal Vector Field

Finally, students calculate the binormal vector field. The direction of the binormal vector is the direction of the head of a pilot. The binormal vector is given by the cross product of the tangent vector and the normal vector, or

$$t1 \times n1.$$

Mathematica code to do this is List5

List5. Cross product

```
Simplify[
  Cross[t1[t], n1[t]], Element[t, Reals]]
```

Again, normalization is performed in the same way as with the tangent vector and the normal vector. An intuitive test for the binormal vector should be carried out at this stage. The binormal vector is defined as follows.

$$\text{b1}[t\_] := \left\{\frac{-2\,\text{Cos}[t] + t\,\text{Sin}[t]}{\sqrt{8+5\,t^2+t^4}}, -\frac{t\,\text{Cos}[t] + 2\,\text{Sin}[t]}{\sqrt{8+5\,t^2+t^4}}, \frac{2+t^2}{\sqrt{8+5\,t^2+t^4}}\right\}$$

Whether tangent, normal, and binormal vectors are correctly calculated can be confirmed by calculating mutual inner products and inner products with themselves. Students should confirm that t_1, n_1 and b_1 form an orthonormal system at each point.

This concludes the computation of the three vectors. Let us display them by 3D graphics of the moving frame as follows.

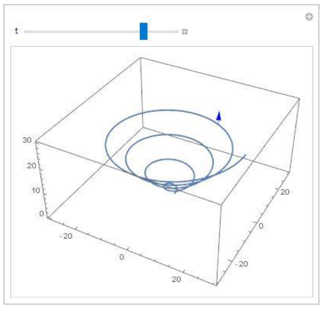

Fig. 4. Binormal vector represented by blue arrow

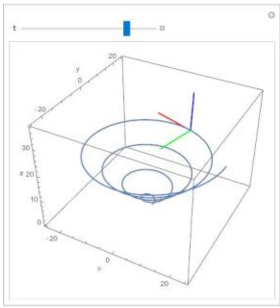

Fig. 5. Unit tangent, normal, and binormal vector

IV. DESIGNING SIMPLE AIRCRAFTS

We give some examples of aircrafts with Mathematica code to generate them. The code in List 6 produces the leftmost one in Fig. 6.

List 6. Aircraft designed

```
FX1000 = {
 White,
  Ball[{1.2, 0, 0.1}, 0.1] (* canopy *),
  Red,
  Triangle[{{1, 0, 0}, {0, 1, 0}, {0, -1,
0}}] (* main wing *),
  Triangle[{{0, 0, 0}, {0.5, 0, 0}, {0,
0, 0.7}}] (* tail wing *),
  Cone[{{0, 0, 0}, {2, 0, 0}}, 0.3]   (*
body *)}
```

V. SAMPLES

In the process so far, students design a curve, calculate various vector fields, and design a flying object. Finally, students make animation by combining these materials. Observing the animation, students can visually learn tangents, normals, and binormals, which are basic concepts in differential geometry of curves.

VI. CONCLUSION

A toolkit was introduced that allows learners to design curves freely and gain geometric insights by creating animation.

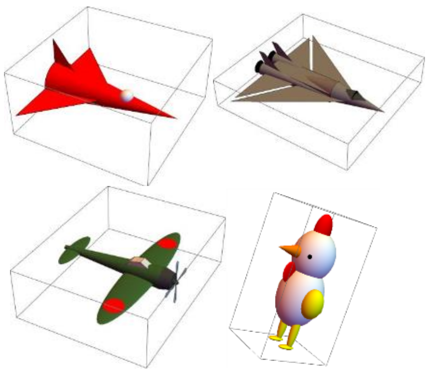

Fig. 6. Example of flying object design

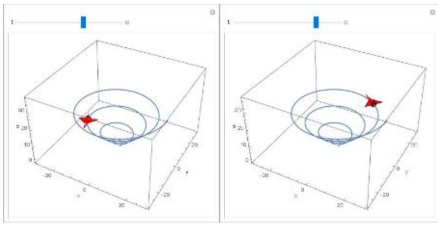

Fig. 7. Flight simulation

VII. CONCLUSION

A toolkit was introduced that allows learners to design curves freely and gain geometric insights by creating animation. This toolkit helps the learner to design and perform mathematical experiments by visualizing key concepts in differential geometry which are difficult to capture intuitively. The toolkit also aims at maintaining the learner's motivation.

Sometimes, exercise problems in differential geometry involving complex calculations deprive learners of their motivation and hide the essence of differential geometry. The basic concepts of differential geometry; tangent, normal, and binormal vector are easily calculated by a computer algebra system with a combination of powerful numerical and graphics facilities.

Presentation materials of this paper are itemized as follows. Mathematical topics taken to demonstrate the patterns are:

- tangent, normal, and binormial vectors,
- cross products.

Focused technical elements specific to Mathematica are [5]:

- dynamical objects in 3d graphics,
- differentiation of curve represented by parameter.

future or on-going work includes

- creating a pool of attracting parts such including new flying objects,
- moving frame techniques, and
- techniques for producing smoother animation.

The authors hope that geometry class will be more attractive to students in engineering departments.

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